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FAMILY OF CONGRUENCES FOR $(2, \beta)-$REGULAR BIPARTITION TRIPLES

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Abstract: Though congruences have their limitations, they have significant importance in the field of number theory and helps in proving many interesting results. Thus, this article has adopted the technique and properties of congruences to identify and prove a set of congruent properties for integer partition. The partition of a positive integer is a way of expressing the number as a sum of positive integers. One such partitions known as regular bipartition triple are discussed in this article. New congruences modulo even integers and modulo prime $(p \geq 5)$ powers are derived for $(2, \beta)-$regular bipartition triples. Also infinite families of congruences modulo 2 for some $(2, \beta)-$regular bipartition triples are derived. The theorems stated in this article are proved using the $q-$series notation and some of the prominent results such as Euler's pentagonal number theorem and Jacobi's triple product identities. There are certain lemmas which are derived using these results that help in proving the major results of this article.

Keywords and Phrases: Integer partitions, Bipartition Triples, Congruences, $q-$series.

2020 Mathematics Subject Classification: 11P81, 11P82, 11P83, 11P84.

1. Introduction

A partition α of an integer $n > 0$ is a non - increasing sequence of positive integers $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ such that, $n = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_k$. $p(n)$ represents the number of partitions of $n > 0$ and its generating function is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1},$$

where for any positive integer a, f_a is defined by

$$f_a = (q^a; q^a)_\infty = \prod_{m=1}^{\infty}(1 - q^{am}), \quad |q| < 1.$$

For an integer $\ell > 1$, a partition is said to be ℓ-regular if none of its parts is divisible by ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n, then the generating function for the number of ℓ-regular partitions of n is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1}.$$

A k-tuple of partitions of n is $(\eta_1, \eta_2, \eta_3, \ldots, \eta_k)$ such that

$$n = |\eta_1| + |\eta_2| + |\eta_3| + \cdots + |\eta_k|.$$

where $|\eta_i|$ is the sum of all parts in partition η_i. A 2-tuple partition is called a bipartition and a 3-tuple partition is called partition triple. An ℓ-regular bipartition of n is an ordered pair (η_1, η_2) of ℓ-regular partition if the sum of all the parts of η_1 and η_2 is equal to n. Let, $B_\ell(n)$ denote the number of ℓ-regular bipartitions of n, then its generating function is given by

$$\sum_{n=0}^{\infty} B_\ell(n)q^n = \frac{f_\ell^2}{f_1^2}.$$

For integers $\alpha, \beta \geq 0$, an (α, β)-regular bipartition of n is a bipartition (η_1, η_2) of n such that η_1 is an α-regular partition and η_2 is a β-regular partition. If $B_{\alpha,\beta}(n)$ denotes the number of (α, β)-regular bipartition of n, then the corresponding generating function is given by

$$\sum_{n=0}^{\infty} B_{\alpha,\beta}(n)q^n = \frac{f_\alpha f_\beta}{f_1^2}.$$

A partition triple (η_1, η_2, η_3) of a positive integer n is called an ℓ-regular partition triple if each of η_1, η_2, η_3 is an ℓ-regular partition. If $BT_\ell(n)$ denotes the number of ℓ-regular partition triple of positive integer n, then the generating function is given by

$$\sum_{n=0}^{\infty} B_\ell(n)q^n = \frac{f_\ell^3}{f_1^3}.$$

If $BT_{\alpha,\beta}(n)$ denotes the number of (α,β)-regular bipartition triples of the positive integer n, then the corresponding generating function is given by

$$\sum_{n=0}^{\infty} BT_{\alpha,\beta}(n)q^n = \frac{f_\alpha^3 f_\beta^3}{f_1^6}. \tag{1.1}$$

Congruences for partitions have been studied well for a long time. The simplest and the most beautiful congruence properties of $p(n), n \geq 0$ are the Ramanujan's congruences

$$\begin{aligned} &p(5n+4) \equiv 0 \pmod{5},\\ &p(7n+5) \equiv 0 \pmod{7}, \text{ and}\\ &p(11n+6) \equiv 0 \pmod{11}. \end{aligned}$$

These congruences and the subsequent study of partition congruences for $p(n)$ have motivated mathematicians to study congruences for special classes of partitions, such as for ℓ-regular partitions, bipartitions, and so on.

Lin [5, 6] proved several infinite families of congruences modulo 3 for $B_{4,4}(n)$ and $B_{7,7}(n)$ and gave characterisations of $B_{4,4}(n)$ modulo 2 and 4. Dai [3] examined the behaviour of $B_{4,4}(n)$ modulo 8 and found several infinite families of congruences modulo 8 for $B_{4,4}(n)$. Dou [4] established infinite family of congruences modulo 11 for $B_{3,11}(n)$. For $\alpha \geq 2$ and $n \geq 0$, Lin [7] proved an infinite family of congruences modulo 3 for 13-regular bipartition of n. The aim of this paper is to study families of congruences modulo even integers and modulo powers of prime $p \geq 5$ for $(2,\beta)$-regular bipartition triples.

The following notations are used in this paper. Ramanujan's general theta function, $f(a,b)$ is given by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

Jacobi's triple product identity is given by

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty,$$

from which we get

$$\Psi(q) = f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

$$f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

In this paper, we prove the following results.

Theorem 1. *For all integers $n > 0$,*

$$BT_{2,3}(3n+1) \equiv 0 \pmod{2}, \tag{1.2}$$
$$BT_{2,3}(3n+2) \equiv 0 \pmod{2}, \tag{1.3}$$
$$BT_{2,3}(9n+4) \equiv 0 \pmod{2}, \tag{1.4}$$
$$BT_{2,3}(9n+7) \equiv 0 \pmod{2}, \tag{1.5}$$
$$BT_{2,3}(3n+2) \equiv 0 \pmod{2^3}, \text{ and} \tag{1.6}$$
$$BT_{2,3}(9n+6) \equiv 0 \pmod{2^3}. \tag{1.7}$$

Theorem 2. *For all integers $n > 0$,*

$$BT_{2,3}(3n+1) \equiv 0 \pmod{6} \text{ and} \tag{1.8}$$
$$BT_{2,3}(3n+2) \equiv 0 \pmod{24}. \tag{1.9}$$

Theorem 3. *For all integers $n \geq 0$ and $\beta \geq 0$,*

$$BT_{2,3}\left(3^{2\beta+1}n + 3\frac{9^\beta - 1}{8}\right) \equiv BT_{2,3}(3n) \pmod{2^2}. \tag{1.10}$$

Theorem 4. *For any prime $p \geq 5$,*

$$f(-q^3) = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{\frac{5k(k+1)}{2}} f\left(-q^{\frac{5p^2+(10k+1)p}{2}}, \quad -q^{\frac{5p^2-(10k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{8}} f(-q^{3p^2}). \tag{1.11}$$

Theorem 5. *For any prime $p \geq 5$, $\beta \geq 0$ and $n \geq 0$,*

$$\sum_{n=0}^{\infty} BT_{2,3}\left(p^{2\beta}n + \frac{p^{2\beta}-1}{24}\right) q^n \equiv f^3(-q^3) \pmod{2}.$$

Theorem 6. *For any prime $p\geq 5$, $\beta\geq 1$, and $n\geq 0$,*

$$BT_{2,3}\left(p^{2\beta}n+\frac{(24i+p)p^{2\beta-1}-1}{24}\right)\equiv 0 \pmod 2, \qquad 1\leq i\leq p-1.$$

Theorem 7. *For any prime $p\geq 5$, $\beta\geq 0$, $n\geq 0$ and $j\in\mathbb{Z}$ such that $1\leq j\leq p-1$ and $\left(\frac{24j+1}{p}\right)=-1$ such that $j\not\equiv\frac{5k^2+k}{2} \pmod p$*

$$BT_{2,3}\left(p^{2\beta+1}n+\frac{(24j+1)p^{2\beta}-1}{24}\right)\equiv 0 \pmod 2.$$

We can generalise the above results as well, to the following.

Theorem 8. *For primes p_i, where $i=1,2,\ldots,r$ with $p_i\geq 5$, $r\geq 0$, and $n\geq 0$,*

$$\sum_{n=0}^{\infty}BT_{2,3}\left(\prod_{s=1}^{r}p_s^2n+\frac{\prod_{s=1}^{r}p_s^2-1}{24}\right)q^n\equiv f^3(-q^3) \pmod 2.$$

Here, $\prod_{s=1}^{0}p_s^2=1$.

Theorem 9. *For primes p_k, $k=1,2,\ldots,r$ with $p_i\geq 5$, $r\geq 1$, and $n\geq 0$,*

$$BT_{2,3}\left(\prod_{s=1}^{r}p_s^2n+\frac{(24i+p_r)\prod_{s=1}^{r-1}p_s^2p_r-1}{24}\right)\equiv 0 \pmod 2, \tag{1.12}$$

where $1\leq i\leq p_r-1$. Also,

$$BT_{2,3}\left(\prod_{s=1}^{r-1}p_s^2p_rn+\frac{(24j+1)\prod_{s=1}^{r-1}p_s^2-1}{24}\right)\equiv 0 \pmod 2, \tag{1.13}$$

where $0\leq j\leq p_r-1$ such that $\left(\frac{24j+1}{p_r}\right)=-1$.

Theorem 10. *For $\beta\geq 3$ and $\delta\in\mathbb{N}$ such that $(12\delta+1)c^2$ is a quadratic non-residue modulo $p,\forall c\in\mathbb{N}$. If there exists some prime $p(\geq 5)|\beta$ then*

$$BT_{2,\beta}(pn+\delta)\equiv 0 \pmod p, \quad \forall n\geq 0.$$

Theorem 11. *If $\beta\geq 3$, $p^r|\beta$ for $p\geq 5$ and $r\geq 2$ then $\forall n\in\mathbb{N}$,*

$$BT_{2,\beta}(pn+\delta)\equiv 0 \pmod{p^2},$$

where $\delta \in \mathbb{N}$ *such that* $(12\delta+1)c^2$ *is a quadratic non-residue modulo* p^2 $\forall\ c \in \mathbb{N}$.

Theorem 12. *If* $\beta \geq 3$, $p^r|\beta$ *for* $p \geq 5$ *and* $r \geq 3$ *then* $\forall n \in \mathbb{N}$,

$$BT_{2,\beta}(pn+\delta) \equiv 0 \pmod{p^3},$$

where $\forall\ \delta.\ c \in \mathbb{N}$ *such that* $(12\delta+1)c^2$ *is a quadratic non-residue modulo* p^3.

The rest of the paper is as follows: in Section 2 we state some preliminary results that we will use to prove the above theorems in Section 3. Finally, we close the paper with some concluding remarks in Section 4.

2. Preliminaries

Euler's pentagonal number theorem gives us

$$(q;q)_\infty = \sum_{m=1}^{\infty} (-1)^m q^{\left(\frac{3}{2}m^2 - \frac{1}{2}m\right)}, \tag{2.14}$$

from which we get

$$(q^p;q^p)_\infty = \sum_{m=1}^{\infty} (-1)^m q^{\left(\frac{3}{2}m^2 - \frac{1}{2}m\right)p}. \tag{2.15}$$

Lemma 1. [1, Lemma 2.2] *The following* 3*-dissections hold true*

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}, \tag{2.16}$$

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9} \tag{2.17}$$

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9} \ \textit{and} \tag{2.18}$$

$$f_1^3 = f_3 a(q^3) - 3q f_9^3, \tag{2.19}$$

where $a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} = 1 + 6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right)$.

Lemma 2. [1] *We have the following from the binomial theorem.*

$$f_k^2 \equiv f_{2k} \pmod{2}, \tag{2.20}$$

$$f_k^4 \equiv f_{2k}^2 \pmod{4}, \tag{2.21}$$

$$f_k^8 \equiv f_{2k}^4 \pmod{8}. \tag{2.22}$$

Lemma 3. [8, Lemma 4.2] *For any non negative integer b and any prime p,*

$$(q^p;q^p)_\infty^b \equiv (q^{bp};q^{bp})_\infty \pmod{p} \tag{2.23}$$

Lemma 4. [2, Equation (1.17)] *For any prime $p\geq 3$ and integer $r\geq 2$,*

$$(q;q)_\infty^{np^r} \equiv (q^p;q^p)_\infty^{np^{r-1}} \pmod{p^2}, \qquad \text{where } n\in\mathbb{N}. \tag{2.24}$$

Lemma 5. [2, Equation (1.19)] *For any prime $p\geq 3$ and integer $r\geq 3$,*

$$(q;q)_\infty^{np^r} \equiv (q^p;q^p)_\infty^{np^{r-1}} \pmod{p^3}, \qquad \text{where } n\in\mathbb{N} \tag{2.25}$$

Remark 1. *For $n\geq 0$, $\sum_{n=0}^{\infty} BT_{2,3}(n)q^n \equiv f^3(-q^3) \pmod{2}$.*

Theorem 13. [9, Theorem 2.1] *For any prime $p\neq 2$,*

$$\Psi(q) = \sum_{k=0}^{\frac{p-3}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, \quad q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\Psi(q^{p^2}).$$

Furthermore, for $0\leq k\leq \frac{p-3}{2}$, $\quad \frac{k^2+k}{2}\not\equiv \frac{p^2-1}{8} \pmod{p}$.

Remark 2. *For any prime $p\geq 5$,*

$$\frac{\pm p-1}{6} \equiv \begin{cases} \frac{p-1}{6}, & p\equiv 1 \pmod{6}, \\ \frac{-p-1}{6}, & p\equiv -1 \pmod{6}. \end{cases}$$

Theorem 14. [9, Theorem 2.2] *For any prime $p\geq 5$,*

$$f(-q) = \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{k(3k+1)}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}).$$

Furthermore, for $\frac{-p-1}{2}\leq k\leq \frac{p-1}{2}$, $\quad \frac{k(3k+1)}{2}\not\equiv \frac{p^2-1}{2} \pmod{p}$.

3. Proof of Theorems

Proof of Theorem 1. Substituting $\alpha=2$ and $\beta=3$ in equation (1.1) we get,

$$\sum_{n=0}^{\infty} BT_{2,3}(n)q^n = \frac{f_2^3 f_3^3}{f_1^6} = \left(\frac{f_2}{f_1^2}\right)^3 f_3^3, \tag{3.26}$$

$$\sum_{n=0}^{\infty} BT_{2,3}(n)q^n \equiv f_3^3 \pmod{2}. \tag{3.27}$$

By extracting the terms containing q^{3n+1}, dividing throughout by q and later replacing q by $q^{\frac{1}{3}}$ in (3.27) we get,

$$\sum_{n=0}^{\infty} BT_{2,3}(3n+1)q^n \equiv 0 \pmod{2}.$$

Similarly, dividing throughout by q^2 for those extracted terms containing q^{3n+2} and then replacing q by $q^{\frac{1}{3}}$ in (3.27) we get,

$$\sum_{n=0}^{\infty} BT_{2,3}(3n+2)q^n \equiv 0 \pmod{2}$$

This gives us the congruence (1.2) and (1.3).

By writing (1.1) modulo 2^2, we get

$$\sum_{n=0}^{\infty} BT_{2,3}(n)q^n \equiv f_3^3 \frac{f_2}{f_1^2} \pmod{2^2}, \tag{3.28}$$

$$\sum_{n=0}^{\infty} BT_{2,3}(n)q^n \equiv \left[\frac{f_3^2 f_6^4 f_9^6}{f_{18}} + 2q f_6 f_9^3\right] \pmod{2^2}. \tag{3.29}$$

By extracting the terms involving the $(3n+2)^{th}$ powers of q and dividing throughout by q and later replacing q by $q^{\frac{1}{3}}$ in (3.29) we get

$$\sum_{n=0}^{\infty} BT_{2,3}(3n+1)q^n \equiv 2 f_2 f_3^3 \pmod{2^2}. \tag{3.30}$$

By extracting the terms involving the $(3n+1)^{th}$ and $(3n+2)^{th}$ powers of q and dividing throughout by q and q^2 respectively and later replacing q by $q^{\frac{1}{3}}$ in (3.30) we arrive at the congruences (1.4) and (1.5).

Furthermore, by writing (1.1) modulo 2^3, we get

$$\sum_{n=0}^{\infty} BT_{2,3}(n)q^n \equiv f_3^3 \left(\frac{f_2}{f_1^2}\right)^3 \pmod{2^3} \equiv \frac{f_6^4 f_9^2}{f_3^5 f_{18}} + 6q \frac{f_6^3 f_{18}^2}{f_3^4 f_9} \pmod{2^3}. \tag{3.31}$$

By extracting the terms involving the $(3n+2)^{th}$ powers of q and dividing throughout by q^2 and later replacing q by $q^{\frac{1}{3}}$ in (3.29) we get the congruence (1.6). Further it follows from (3.31) that

$$\sum_{n=0}^{\infty} BT_{2,3}(3n)q^n \equiv \frac{f_2^4 f_3^3}{f_1^5 f_6} \pmod{2^3} \equiv \frac{f_3^2}{f_6}\left(f_3\ a(q^3) - 3q f_9^3\right) \pmod{2^3}. \tag{3.32}$$

By extracting the terms involving the $(3n+2)^{th}$ powers of q and dividing throughout by q^2 and later replacing q by $q^{\frac{1}{3}}$ in (3.32) we arrive at the congruence (1.7).

Remark 3. *From the above proof, we get more generally,*

$$\sum_{n=0}^{\infty} BT_{2,3k}(n)q^n \equiv f_{3k}^3 \pmod 2.$$

So, congruences (1.2) and (1.3) are true for $BT_{2,3k}(3n+1)$ *and* $BT_{2,3k}(3n+2)$ *respectively.*

The proof of Theorem 2 is exactly similar to the above proof, so for the sake of brevity we omit the details here.

Proof of Theorem 3. Extracting $3n$ from (3.28) and applying (2.20) we get,

$$\sum_{n=0}^{\infty} BT_{2,3}(3n)q^n \equiv \frac{f_2^2 f_3^2}{f_1 f_6} \pmod 4, \tag{3.33}$$

$$\equiv \frac{f_3 f_9^2}{f_{18}} + q\frac{f_{18}^2 f_3^2}{f_9 f_6} \pmod 4. \tag{3.34}$$

By extracting the terms involving $(3n+1)^{th}$ powers of q in (3.34) we have,

$$\sum_{n=0}^{\infty} BT_{2,3}(9n+3)q^n \equiv \frac{f_6^2 f_1^2}{f_3 f_2} \pmod 4, \tag{3.35}$$

$$\equiv \frac{f_6^2 f_9^2}{f_3 f_{18}} + 2q\frac{f_3 f_{18}^2}{f_6 f_9} \pmod 4. \tag{3.36}$$

By extracting the terms involving $(3n)^{th}$ powers of q in (3.36) we have,

$$\sum_{n=0}^{\infty} BT_{2,3}(27n+3)q^n \equiv \frac{f_2^2 f_3^2}{f_1 f_6} \pmod 4 \equiv \sum_{n=0}^{\infty} BT_{2,3}(3n)q^n \pmod 4 \tag{3.37}$$

Now by iterating n in (3.34) by $9n+1$ we arrive at the congruence (1.10).

Proof of Theorem 4.

$$
\begin{aligned}
f(-q^3) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(5n+1)}{2}}\\
&= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}}\sum_{n=-\infty}^{\infty}(-1)^{pn+k}q^{\frac{5p^2n^2+10pnk+5k^2+pn+k}{2}}\\
&= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}}(-1)^k q^{\frac{k(5k+1)}{2}}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5p^2n^2+(10k+1)pn}{2}}\\
&= \sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}}(-1)^k q^{\frac{k(5k+1)}{2}}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5p^2n^2+(10k+1)pn}{2}}\\
&\quad + (-1)^{\frac{\pm p-1}{6}}q^{\frac{p^2-1}{8}}\sum_{n=-\infty}^{\infty}(-1)^n q^{3p^2\frac{n(5n+1)}{2}}\\
&= \sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}}(-1)^k q^{\frac{k(5k+1)}{2}}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5p^2n^2+(10k+1)pn}{2}} + (-1)^{\frac{\pm p-1}{6}}q^{\frac{p^2-1}{8}}f(-q^{3p^2})\\
&= \sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{\frac{5k(k+1)}{2}}f\left(-q^{\frac{5p^2+(10k+1)p}{2}},\quad -q^{\frac{5p^2-(10k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}}q^{\frac{p^2-1}{8}}f(-q^{3p^2})
\end{aligned}
$$

Proof of Theorem 5. We prove the theorem by induction on β. When $\beta = 0$ we arrive at Remark 1. Suppose the above theorem holds true for β. Then using Theorem 4, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} BT_{2,3}\left(p^{2\beta}\left(pn+\frac{p^2-1}{24}\right)+\frac{p^{2\beta}-1}{24}\right)q^n &= \sum_{n=0}^{\infty} BT_{2,3}\left(p^{2\beta+1}n+\frac{p^{2\beta+2}-1}{24}\right)q^n\\
&\equiv f^3(-q^{3p}) \pmod 2. \qquad (3.38)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} BT_{2,3}\left(p^{2\beta}\left(p^2n+\frac{p^2-1}{24}\right)+\frac{p^{2\beta}-1}{24}\right)q^n &= \sum_{n=0}^{\infty} BT_{2,3}\left(p^{2\beta+2}n+\frac{p^{2\beta+2}-1}{24}\right)q^n\\
&\equiv f^3(-q^3) \pmod 2. \qquad (3.39)
\end{aligned}
$$

Therefore, the theorem holds true for $\beta+1$.

Proof of Theorem 6. By using (3.38) for the values of $i=1,2,3,\ldots,p-1$ we get,

$$BT_{2,3}\left(p^{2\beta+1}(pn+i)+\frac{p^{2\beta+2}-1}{24}\right)\equiv 0 \pmod 2. \tag{3.40}$$

Proof of Theorem 7. Since we have $j\not\equiv \frac{5k^2+k}{2} \pmod p$ for $|k|\le \frac{p-1}{2}$, with the help of Theorems 4 and 5 it can be shown that

$$BT_{2,3}\left(p^{2\beta}(pn+j)+\frac{p^{2\beta}-1}{24}\right)\equiv 0 \pmod 2.$$

Proof of Theorem 8. We prove the theorem by induction on r. From theorem 5, for $r=0$ we have

$$\sum_{n=0}^{\infty} BT_{2,3}\left(\prod_{s=1}^{0} p_s^2 n+\frac{\prod_{s=1}^{0} p_s^2-1}{24}\right)q^n\equiv f^3(-q^3) \pmod 2.$$

Now, from Theorem 4, for a prime $p_{r+1}\ge 5$ we have

$$\begin{aligned}&\sum_{n=0}^{\infty} BT_{2,3}\left(\prod_{s=1}^{r} p_s^2\left(p_{r+1}^2 n+\frac{p_{r+1}^2-1}{24}\right)+\frac{\prod_{s=1}^{r} p_s^2-1}{24}\right)q^n\\ &=\sum_{n=0}^{\infty} BT_{2,3}\left(\prod_{s=1}^{r+1} p_s^2 n+\frac{\prod_{s=1}^{r+1} p_s^2-1}{24}\right)q^n\\ &\equiv f^3(-q^3) \pmod 2.\end{aligned}$$

This completes the proof.

Proof of Theorem 9. From application of Theorems 4 and 8 we get equation (1.12). The proof of equation (1.13) is similar to the proof of Theorem 7, so we omit it here.

Proof of Theorem 10. From (1.1), we have

$$\sum_{n=0}^{\infty} BT_{2,\beta}(n)q^n=\frac{(q^2;q^2)^3\left(q^\beta;q^\beta\right)^3}{(q;q)^6}. \tag{3.41}$$

Since $p|\beta$ implies $\beta=m_1p$ for some $m_1\in\mathbb{N}$. Hence by incorporating Lemma 3 we get,

$$\sum_{n=0}^{\infty} BT_{2,\beta}(n)q^n=\frac{(q^2;q^2)^3\,(q^p;q^p)^{3m_1}}{(q;q)^6} \pmod p. \tag{3.42}$$

With the help of Euler's pentagonal number theorem (2.14) and (2.15), the above equation can be reduced to

$$\sum_{n=0}^{\infty} BT_{2,\beta}(n)q^n = \left[\sum_{t_1=1}^{\infty}(-1)^{t_1}q^{2\left(\frac{3}{2}t_1^2-\frac{1}{2}t_1\right)}\right]^3 \left[\sum_{t_2=1}^{\infty}(-1)^{t_2}q^{p^r\left(\frac{3}{2}t_2^2-\frac{1}{2}t_2\right)}\right]^{3m_1} \left[\sum_{l=1}^{\infty}p(l)q^l\right]^3 \pmod p. \tag{3.43}$$

Suppose there exists $t_1,\ t_2$ and l such that the powers of q add up to $pn+\delta$ then

$$\begin{aligned} \delta &\equiv 3t_1^2 - t_1 \pmod p,\\ 12\delta+1 &\equiv 36t_1^2-12t_1+1 \pmod p,\\ (12\delta+1)c^2 &\equiv (36t_1^2-12t_1+1)c^2 \pmod p,\\ (12\delta+1)c^2 &\equiv [(6t_1+1)c]^2 \pmod p. \end{aligned} \tag{3.44}$$

The congruence (3.44) contradicts the fact that $(12\delta+1)c^2$ is a quadratic non residue modulo p. This proves the theorem.

The proofs of Theorems 11 and 12 are exactly similar to the proof of Theorem 10, but with an application of Lemmas 4 and 5 respectively, so we omit the details here.

4. Concluding Remarks

We have only done a basic study of the possible congruences for $BT_{2,3}(n)$. It is hoped that more general results can be found. For instance, experiments suggest that congruences similar to the ones stated in Theorem 1 are true for $BT_{2,6}$ and $BT_{2,9}$. This suggests the following open questions.

Question 1. *Can the congruences in Theorems 1 and 2 be generalized for* $BT_{2,3k}(n)$, *for* $n \geq 0$ *and* $k > 1$*?*

Question 2. *Can the results of the type of Theorems 5-9 be extended for other cases of* $BT_{\alpha,\beta}(n)$, *for* $n \geq 0$*?*

We close the paper with the following conjecture.

Conjecture 1. *For all* $n \geq 0$ *and* $k > 1$, *we have* $BT_{2,3k}(kn) \equiv BT_{2,3}(n) \pmod 2$.

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